

# Auxiliary results for “Nonparametric kernel estimation of the probability density function of regression errors using estimated residuals”

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This version: March 2012

Let  $(X_1, Y_1), \dots, (X_n, Y_n)$  be a sample of independent replicates of the random vector  $(X, Y)$ , where  $Y$  is the univariate dependent variable and  $X$  is the covariate of dimension  $d$ . Let  $m(\cdot)$  be the conditional expectation of  $Y$  given  $X$  and let  $\varepsilon$  be the related regression error term, so that the regression model is

$$Y = m(X) + \varepsilon,$$

where  $\varepsilon$  is assumed to have mean zero and to be statistically independent of  $X$ , and the function  $m$  is smooth but unknown. Our aim is to investigate the nonparametric estimation of the probability density function of the error term  $\varepsilon$ .

## 1 Construction of the estimator

Define

$$\hat{\varepsilon}_i = Y_i - \hat{m}_{in}, \quad i = 1, \dots, n,$$

where  $\hat{m}_{in} = \hat{m}_{in}(X_i)$  is the leave-one out version of the Nadaraya-Watson (1964) kernel estimator of  $m(X_i)$ ,

$$\hat{m}_{in} = \frac{\sum_{\substack{j=1 \\ j \neq i}}^n Y_j K_0 \left( \frac{X_j - X_i}{b_0} \right)}{\sum_{\substack{j=1 \\ j \neq i}}^n K_0 \left( \frac{X_j - X_i}{b_0} \right)}.$$

Here  $K_0(\cdot)$  is a kernel function defined on  $\mathbb{R}^d$  and  $b_0 = b_0(n)$  is a bandwidth sequence.

The proposed estimator for the density  $f$  of  $\varepsilon$  is

$$\hat{f}_n(e) = \frac{1}{b_1 \sum_{i=1}^n \mathbb{1}(X_i \in \mathcal{X}_0)} \sum_{i=1}^n \mathbb{1}(X_i \in \mathcal{X}_0) K_1 \left( \frac{\hat{\varepsilon}_i - e}{b_1} \right), \quad e \in \mathbb{R},$$

where  $\mathcal{X}_0$  is an inner subset of the support  $\mathcal{X}$  of the covariate  $X$ ,  $K_1(\cdot)$  is a univariate kernel function and  $b_1 = b_1(n)$  is a bandwidth sequence.

## 2 Assumptions

(**A<sub>1</sub>**) The support  $\mathcal{X}$  of  $X$  is a subset of  $\mathbb{R}^d$ ,  $\mathcal{X}_0$  has a nonempty interior and the closure of  $\mathcal{X}_0$  is in the interior of  $\mathcal{X}$ .

(**A<sub>2</sub>**) The p.d.f.  $g(\cdot)$  of the i.i.d. covariates  $X_i$  is strictly positive over the closure of  $\mathcal{X}_0$ , and has continuous second order partial derivatives over  $\mathcal{X}$ .

(**A<sub>3</sub>**) The regression function  $m(\cdot)$  has continuous second order partial derivatives over  $\mathcal{X}$ .

(**A<sub>4</sub>**) The i.i.d. centered error regression terms  $\varepsilon_i$  have finite 6th moments and are independent of the covariates  $X_i$ .

(**A<sub>5</sub>**) The probability density function  $f(\cdot)$  of the  $\varepsilon_i$ 's has bounded continuous second order derivatives over  $\mathbb{R}$  and satisfies  $\sup_{e \in \mathbb{R}} |h_p^{(k)}(e)| < \infty$ , where  $h_p(e) = e^p f(e)$ ,  $p \in [0, 2]$  and  $k \in \{0, 1, 2\}$ .

(**A<sub>6</sub>**) The kernel function  $K_0$  is symmetric, continuous over  $\mathbb{R}^d$  with support contained in  $[-1/2, 1/2]^d$  and satisfies  $\int K_0(z) dz = 1$ .

(**A<sub>7</sub>**) The kernel function  $K_1$  is symmetric, has a compact support, is three times continuously differentiable over  $\mathbb{R}$ , and satisfies  $\int K_1(v) dv = 1$ ,  $\int K_1^{(\ell)}(v) dv = 0$  for  $\ell = 1, 2, 3$ , and  $\int v K_1^{(\ell)}(v) dv = 0$  for  $\ell = 2, 3$ .

(**A<sub>8</sub>**) The bandwidth  $b_0$  decreases to 0 when  $n \rightarrow \infty$  and satisfies, for  $d^* = \sup\{d + 2, 2d\}$ ,  $nb_0^{d^*} / \ln n \rightarrow \infty$  and  $\ln(1/b_0) / \ln(\ln n) \rightarrow \infty$  when  $n \rightarrow \infty$ .

(**A<sub>9</sub>**) The bandwidth  $b_1$  decreases to 0 and satisfies  $n^{(d+8)} b_1^{7(d+4)} \rightarrow \infty$  when  $n \rightarrow \infty$ .

### 3 Auxiliary results

**Proposition 3.1.** *Define*

$$\beta_{in} = \frac{\mathbb{1}(X_i \in \mathcal{X}_0)}{nb_0^d \widehat{g}_{in}} \sum_{j=1, j \neq i}^n (m(X_j) - m(X_i)) K_0 \left( \frac{X_j - X_i}{b_0} \right),$$

where

$$\widehat{g}_{in} = \frac{1}{nb_0^d} \sum_{j=1, j \neq i}^n K_0 \left( \frac{X_j - X_i}{b_0} \right).$$

Then, under  $(A_1) - (A_9)$ , we have, for all  $e \in \mathbb{R}$  and  $b_0$  and  $b_1$  go to 0,  $\sup_i |\beta_{in}| = O_{\mathbb{P}}(b_0^2)$  and

$$\sum_{i=1}^n \beta_{in} K_1^{(1)} \left( \frac{\varepsilon_i - e}{b_1} \right) = O_{\mathbb{P}}(b_0^2) (nb_1^2 + (nb_1)^{1/2}).$$

**Proposition 3.2.** *Set*

$$\Sigma_{in} = \frac{\mathbb{1}(X_i \in \mathcal{X}_0)}{nb_0^d \widehat{g}_{in}} \sum_{j=1, j \neq i}^n \varepsilon_j K_0 \left( \frac{X_j - X_i}{b_0} \right).$$

Then, under  $(A_1) - (A_9)$ , we have, for all  $e \in \mathbb{R}$  and  $b_0$  and  $b_1$  going to 0,

$$\sum_{i=1}^n \Sigma_{in} K_1^{(1)} \left( \frac{\varepsilon_i - e}{b_1} \right) = O_{\mathbb{P}} \left( nb_1^4 + \frac{b_1}{b_0^d} \right)^{1/2}.$$

**Proposition 3.3.** *Let  $\text{Var}_n(\cdot)$  be the conditional variance given  $X_1, \dots, X_n$ , and set*

$$\zeta_{in} = \mathbb{1}(X_i \in \mathcal{X}_0) (\widehat{m}_{in} - m(X_i))^2 K_1^{(2)} \left( \frac{\varepsilon_i - e}{b_1} \right).$$

Then under  $(A_1) - (A_9)$ , we have, for all  $e \in \mathbb{R}$ , and  $b_0$  and  $b_1$  going to 0,

$$\text{Var}_n \left( \sum_{i=1}^n \zeta_{in} \right) = O_{\mathbb{P}} \left( nb_1 + n^2 b_0^d b_1^{7/2} \right) \left( b_0^4 + \frac{1}{nb_0^d} \right)^2.$$

**Proposition 3.4.** *Let  $R_{in} = \mathbb{1}(X_i \in \mathcal{X}_0) (\widehat{m}_{in} - m(X_i))^3 I_{in}$ , where*

$$I_{in} = \int_0^1 (1-t)^2 K_1^{(3)} \left( \frac{\varepsilon_i - t(\widehat{m}_{in} - m(X_i)) - e}{b_1} \right) dt.$$

Then, under  $(A_1) - (A_9)$ , we have, for all  $e \in \mathbb{R}$ , and  $b_0$  and  $b_1$  going to 0,

$$\text{Var}_n \left( \sum_{i=1}^n R_{in} \right) = O_{\mathbb{P}} \left( n^2 b_0^d b_1 \right) \left( b_0^4 + \frac{1}{n b_0^d} \right)^3.$$

## 4 Intermediate results for the propositions

The proofs of the propositions are based on the following results.

**Lemma 4.1.** *Define*

$$\widehat{g}_n(x) = \frac{1}{n b_0^d} \sum_{i=1}^n K_0 \left( \frac{X_i - x}{b_0} \right), \quad \overline{g}_n(x) = \mathbb{E} [\widehat{g}_n(x)], \quad x \in \mathcal{X}_0.$$

Then under  $(A_1) - (A_2)$ ,  $(A_6)$  and  $(A_8)$ , we have, when  $b_0$  goes to 0,

$$\sup_{x \in \mathcal{X}_0} |\overline{g}_n(x) - g(x)| = O(b_0^2), \quad \sup_{x \in \mathcal{X}_0} |\widehat{g}_n(x) - \overline{g}_n(x)| = O_{\mathbb{P}} \left( b_0^4 + \frac{\ln n}{n b_0^d} \right)^{1/2},$$

and

$$\sup_{x \in \mathcal{X}_0} \left| \frac{1}{\widehat{g}_n(x)} - \frac{1}{g(x)} \right| = O_{\mathbb{P}} \left( b_0^4 + \frac{\ln n}{n b_0^d} \right)^{1/2}.$$

**Lemma 4.2.** *Assume that  $(A_4)$  and  $(A_6)$  hold. Then, for any  $1 \leq i \neq j \leq n$ ,*

$$(\widehat{m}_{in} - m(X_i), \varepsilon_i) \text{ and } (\widehat{m}_{jn} - m(X_j), \varepsilon_j)$$

*are independent given  $X_1, \dots, X_n$ , provided that  $\|X_i - X_j\| \geq C b_0$ , for some  $C > 0$ .*

**Lemma 4.3.** *Let  $\mathbb{E}_n[\cdot]$  be the conditional mean given  $X_1, \dots, X_n$ , and assume  $(A_1) - (A_9)$ . Then,*

$$\begin{aligned} \sup_{1 \leq i \leq n} \mathbb{E}_n [\mathbb{1}(X_i \in \mathcal{X}_0) (\widehat{m}_{in} - m(X_i))^4] &= O_{\mathbb{P}} \left( b_0^4 + \frac{1}{n b_0^d} \right)^2, \\ \sup_{1 \leq i \leq n} \mathbb{E}_n [\mathbb{1}(X_i \in \mathcal{X}_0) (\widehat{m}_{in} - m(X_i))^6] &= O_{\mathbb{P}} \left( b_0^4 + \frac{1}{n b_0^d} \right)^3. \end{aligned}$$

**Lemma 4.4.** Under (A<sub>5</sub>) and (A<sub>7</sub>) we have, for some  $C > 0$ , and for any  $e \in \mathbb{R}$  and  $p \in [0, 2]$ ,

$$\left| \int K_1^{(1)} \left( \frac{\epsilon - e}{b_1} \right)^2 \epsilon^p f(\epsilon) d\epsilon \right| \leq Cb_1, \quad \left| \int K_1^{(1)} \left( \frac{\epsilon - e}{b_1} \right) \epsilon^p f(\epsilon) d\epsilon \right| \leq Cb_1^2, \quad (4.1)$$

$$\left| \int K_1^{(2)} \left( \frac{\epsilon - e}{b_1} \right)^2 \epsilon^p f(\epsilon) d\epsilon \right| \leq Cb_1, \quad \left| \int K_1^{(2)} \left( \frac{\epsilon - e}{b_1} \right) \epsilon^p f(\epsilon) d\epsilon \right| \leq Cb_1^3, \quad (4.2)$$

$$\left| \int K_1^{(3)} \left( \frac{\epsilon - e}{b_1} \right)^2 \epsilon^p f(\epsilon) d\epsilon \right| \leq Cb_1, \quad \left| \int K_1^{(3)} \left( \frac{\epsilon - e}{b_1} \right) \epsilon^p f(\epsilon) d\epsilon \right| \leq Cb_1^3. \quad (4.3)$$

The proof of all these lemmas is postponed in the appendix.

## 5 Proofs of the auxiliary results

### Proof of Proposition 3.1

Assumption (A<sub>4</sub>) and Lemma 4.4-(4.1) yield

$$\begin{aligned} \left| \mathbb{E}_n \left[ \sum_{i=1}^n \beta_{in} K_1^{(1)} \left( \frac{\varepsilon_i - e}{b_1} \right) \right] \right| &= \left| \mathbb{E} \left[ K_1^{(1)} \left( \frac{\varepsilon - e}{b_1} \right) \right] \sum_{i=1}^n \beta_{in} \right| \leq Cnb_1^2 \max_{1 \leq i \leq n} |\beta_{in}|, \\ \text{Var}_n \left[ \sum_{i=1}^n \beta_{in} K_1^{(1)} \left( \frac{\varepsilon_i - e}{b_1} \right) \right] &\leq \sum_{i=1}^n \beta_{in}^2 \mathbb{E} \left[ K_1^{(1)} \left( \frac{\varepsilon - e}{b_1} \right)^2 \right] \leq Cnb_1 \max_{1 \leq i \leq n} |\beta_{in}|^2. \end{aligned}$$

Hence the (conditional) Markov inequality gives

$$\sum_{i=1}^n \beta_{in} K_1^{(1)} \left( \frac{\varepsilon_i - e}{b_1} \right) = O_{\mathbb{P}} \left( nb_1^2 + (nb_1)^{1/2} \right) \max_{1 \leq i \leq n} |\beta_{in}|,$$

so that the proposition follows if we can prove that

$$\sup_{1 \leq i \leq n} |\beta_{in}| = O_{\mathbb{P}} (b_0^2), \quad (5.1)$$

as established now. For this, define

$$\zeta_j(x) = \mathbb{1}(x \in \mathcal{X}_0) (m(X_j) - m(x)) K_0 \left( \frac{X_j - x}{b_0} \right), \quad \nu_{in}(x) = \frac{1}{(n-1)b_0^d} \sum_{j=1, j \neq i}^n (\zeta_j(x) - \mathbb{E}[\zeta_j(x)]),$$

and  $\bar{\nu}_n(x) = \mathbb{E}[\zeta_j(x)]/b_0^d$ , so that

$$\beta_{in} = \frac{n-1}{n} \frac{\nu_{in}(X_i) + \bar{\nu}_n(X_i)}{\hat{g}_{in}}.$$

For  $\max_{1 \leq i \leq n} |\bar{\nu}_n(X_i)|$ , first observe that a second-order Taylor expansion applied successively to  $g(\cdot)$  and  $m(\cdot)$  give, for  $b_0$  small enough, and for any  $x, z$  in  $\mathcal{X}$ ,

$$\begin{aligned} & [m(x + b_0 z) - m(x)] g(x + b_0 z) \\ &= \left[ b_0 m^{(1)}(x) z + \frac{b_0^2}{2} z m^{(2)}(x + \theta_1 b_0 z) z^\top \right] \left[ g(x) + b_0 g^{(1)}(x) z + \frac{b_0^2}{2} z g^{(2)}(x + \theta_2 b_0 z) z^\top \right], \end{aligned}$$

for some  $\theta_1 = \theta_1(x, b_0 z)$  and  $\theta_2 = \theta_2(x, b_0 z)$  in  $[0, 1]$ . Therefore, since  $\int z K(z) dz = 0$  under  $(A_7)$ , it follows that, by  $(A_1)$ ,  $(A_2)$  and  $(A_3)$ ,

$$\begin{aligned} \max_{1 \leq i \leq n} |\bar{\nu}_n(X_i)| &\leq \sup_{x \in \mathcal{X}_0} |\bar{\nu}_n(x)| = \sup_{x \in \mathcal{X}_0} \left| \int (m(x + b_0 z) - m(x)) K_0(z) g(x + b_0 z) dz \right| \\ &\leq C b_0^2. \end{aligned} \tag{5.2}$$

Consider now the term  $\max_{1 \leq i \leq n} |\nu_{in}(X_i)|$ . The Bernstein inequality (see e.g. Serfling (2002)) and  $(A_4)$  give, for any  $t > 0$ ,

$$\begin{aligned} \mathbb{P} \left( \max_{1 \leq i \leq n} |\nu_{in}(X_i)| \geq t \right) &\leq \sum_{i=1}^n \mathbb{P} (|\nu_{in}(X_i)| \geq t) \leq \sum_{i=1}^n \int \mathbb{P} (|\nu_{in}(x)| \geq t | X_i = x) g(x) dx \\ &\leq 2n \exp \left( - \frac{(n-1)t^2}{2 \sup_{x \in \mathcal{X}_0} \text{Var}(\zeta_j(x)/b_0^d) + \frac{4M}{3b_0^d} t} \right), \end{aligned}$$

where  $M$  is such that  $\sup_{x \in \mathcal{X}_0} |\zeta_j(x)| \leq M$ . The definition of  $\mathcal{X}_0$  given in  $(A_2)$ ,  $(A_3)$ ,  $(A_7)$  and the standard Taylor expansion yield, for  $b_0$  small enough,

$$\sup_{x \in \mathcal{X}_0} |\zeta_j(x)| \leq C b_0, \quad \sup_{x \in \mathcal{X}_0} \text{Var}(\zeta_j(x)/b_0^d) \leq \frac{1}{b_0^d} \sup_{x \in \mathcal{X}_0} \int (m(x + b_0 z) - m(x))^2 K_0^2(z) g(x + b_0 z) dz \leq \frac{C b_0^2}{b_0^d},$$

so that, for any  $t \geq 0$ ,

$$\mathbb{P} \left( \max_{1 \leq i \leq n} |\nu_{in}(X_i)| \geq t \right) \leq 2n \exp \left( - \frac{(n-1)b_0^d t^2 / b_0^2}{C + C t / b_0} \right).$$

This gives

$$\mathbb{P} \left( \max_{1 \leq i \leq n} |\nu_{in}(X_i)| \geq \left( \frac{b_0^2 \ln n}{(n-1)b_0^d} \right)^{1/2} t \right) \leq 2n \exp \left( - \frac{t^2 \ln n}{C + C t \left( \frac{\ln n}{(n-1)b_0^d} \right)^{1/2}} \right) = o(1),$$

provided that  $t$  is large enough and under  $(A_9)$ . It then follows that

$$\max_{1 \leq i \leq n} |\nu_{in}(X_i)| = O_{\mathbb{P}} \left( \frac{b_0^2 \ln n}{n b_0^d} \right)^{1/2}.$$

This order, (5.2) and Lemma 4.1 show that (5.1) is proved, since  $(b_0^2 \ln n / (nb_0^d))^{1/2} = O(b_0^2)$  under  $(A_9)$ , and that

$$\beta_{in} = \frac{n-1}{n} \frac{\nu_{in}(X_i) + \bar{\nu}_n(X_i)}{\widehat{g}_{in}}. \square$$

### Proof of Proposition 3.2

Assumption  $(A_4)$  implies that  $\Sigma_{in}$  is independent of  $\varepsilon_i$ , and that  $\mathbb{E}_n[\Sigma_{in}] = 0$ . This yields

$$\mathbb{E}_n \left[ \sum_{i=1}^n \Sigma_{in} K_1^{(1)} \left( \frac{\varepsilon_i - e}{b_1} \right) \right] = 0. \quad (5.3)$$

Moreover, observe that

$$\begin{aligned} & \text{Var}_n \left[ \sum_{i=1}^n \Sigma_{in} K_1^{(1)} \left( \frac{\varepsilon_i - e}{b_1} \right) \right] \\ &= \sum_{i=1}^n \text{Var}_n \left[ \Sigma_{in} K_1^{(1)} \left( \frac{\varepsilon_i - e}{b_1} \right) \right] + \sum_{i=1}^n \sum_{\substack{j=1 \\ j \neq i}}^n \text{Cov}_n \left[ \Sigma_{in} K_1^{(1)} \left( \frac{\varepsilon_i - e}{b_1} \right), \Sigma_{jn} K_1^{(1)} \left( \frac{\varepsilon_j - e}{b_1} \right) \right]. \end{aligned} \quad (5.4)$$

By Lemma 4.4(4.1) and  $(A_4)$ , the first term above gives

$$\begin{aligned} \sum_{i=1}^n \text{Var}_n \left[ \Sigma_{in} K_1^{(1)} \left( \frac{\varepsilon_i - e}{b_1} \right) \right] &\leq \sum_{i=1}^n \mathbb{E}_n [\Sigma_{in}^2] \mathbb{E} \left[ K_1^{(1)} \left( \frac{\varepsilon_i - e}{b_1} \right)^2 \right] \\ &\leq \frac{Cb_1\sigma^2}{(nb_0^d)^2} \sum_{i=1}^n \sum_{\substack{j=1 \\ j \neq i}}^n \frac{\mathbb{1}(X_i \in \mathcal{X}_0)}{\widehat{g}_{in}^2} K_0^2 \left( \frac{X_j - X_i}{b_0} \right) \\ &\leq \frac{Cb_1\sigma^2}{nb_0^d} \sum_{i=1}^n \frac{\mathbb{1}(X_i \in \mathcal{X}_0) \widetilde{g}_{in}}{\widehat{g}_{in}^2}, \end{aligned} \quad (5.5)$$

where  $\sigma^2 = \text{Var}(\varepsilon)$  and

$$\widetilde{g}_{in} = \frac{1}{nb_0^d} \sum_{j=1, j \neq i}^n K_0^2 \left( \frac{X_j - X_i}{b_0} \right).$$

For the sum of conditional covariances in (5.4), write

$$\begin{aligned} & \sum_{i=1}^n \sum_{\substack{j=1 \\ j \neq i}}^n \text{Cov}_n \left[ \Sigma_{in} K_1^{(1)} \left( \frac{\varepsilon_i - e}{b_1} \right), \Sigma_{jn} K_1^{(1)} \left( \frac{\varepsilon_j - e}{b_1} \right) \right] \\ &= \sum_{i=1}^n \sum_{\substack{j=1 \\ j \neq i}}^n \mathbb{E}_n \left[ \Sigma_{in} \Sigma_{jn} K_1^{(1)} \left( \frac{\varepsilon_i - e}{b_1} \right) K_1^{(1)} \left( \frac{\varepsilon_j - e}{b_1} \right) \right] \\ &= \sum_{i=1}^n \sum_{\substack{j=1 \\ j \neq i}}^n \frac{\mathbb{1}(X_i \in \mathcal{X}_0) \mathbb{1}(X_j \in \mathcal{X}_0)}{(nb_0^d)^2 \widehat{g}_{in} \widehat{g}_{jn}} \sum_{\substack{k=1 \\ k \neq i}}^n \sum_{\substack{\ell=1 \\ \ell \neq j}}^n K_0 \left( \frac{X_k - X_i}{b_0} \right) K_0 \left( \frac{X_\ell - X_j}{b_0} \right) \mathbb{E} [\xi_{ki} \xi_{\ell j}], \end{aligned}$$

where  $\xi_{ki} = \varepsilon_k K_1^{(1)}((\varepsilon_i - e)/b_1)$ . Moreover, under  $(A_4)$ , it is seen that for  $k \neq \ell$ ,  $\mathbb{E}[\xi_{ki}\xi_{\ell j}] = 0$  when  $\text{Card}\{i, j, k, \ell\} \geq 3$ . Therefore the symmetry of  $K_0$  implies that

$$\begin{aligned} & \sum_{i=1}^n \sum_{\substack{j=1 \\ j \neq i}}^n \text{Cov}_n \left[ \Sigma_{in} K_1^{(1)} \left( \frac{\varepsilon_i - e}{b_1} \right), \Sigma_{jn} K_1^{(1)} \left( \frac{\varepsilon_j - e}{b_1} \right) \right] \\ &= \sum_{i=1}^n \sum_{\substack{j=1 \\ j \neq i}}^n \frac{\mathbb{1}(X_i \in \mathcal{X}_0) \mathbb{1}(X_j \in \mathcal{X}_0)}{(nb_0^d)^2 \widehat{g}_{in} \widehat{g}_{jn}} K_0^2 \left( \frac{X_j - X_i}{b_0} \right) \mathbb{E}^2 \left[ \varepsilon K_1^{(1)} \left( \frac{\varepsilon - e}{b_1} \right) \right] \\ &+ \sum_{i=1}^n \sum_{\substack{j=1 \\ j \neq i}}^n \frac{\mathbb{1}(X_i \in \mathcal{X}_0) \mathbb{1}(X_j \in \mathcal{X}_0)}{(nb_0^d)^2 \widehat{g}_{in} \widehat{g}_{jn}} \sum_{\substack{k=1 \\ k \neq i, j}}^n K_0 \left( \frac{X_k - X_i}{b_0} \right) K_0 \left( \frac{X_k - X_j}{b_0} \right) \mathbb{E}[\varepsilon^2] \mathbb{E}^2 \left[ K_1^{(1)} \left( \frac{\varepsilon - e}{b_1} \right) \right]. \end{aligned}$$

Hence from Lemma 4.1 and Lemma 4.4-(4.1), we deduce

$$\begin{aligned} & \left| \sum_{i=1}^n \sum_{\substack{j=1 \\ j \neq i}}^n \text{Cov}_n \left[ \Sigma_{in} K_1^{(1)} \left( \frac{\varepsilon_i - e}{b_1} \right), \Sigma_{jn} K_1^{(1)} \left( \frac{\varepsilon_j - e}{b_1} \right) \right] \right| \\ &= O_{\mathbb{P}} \left( \frac{b_1^4}{nb_0^d} \right) \sum_{i=1}^n \frac{\mathbb{1}(X_i \in \mathcal{X}_0) \widetilde{g}_{in}}{\widehat{g}_{in}} + O_{\mathbb{P}}(b_1^4) \sum_{i=1}^n \frac{\mathbb{1}(X_i \in \mathcal{X}_0) g_{in}}{\widehat{g}_{in}}, \end{aligned}$$

where  $\widetilde{g}_{in}$  is defined as in (5.4) and

$$g_{in} = \frac{1}{(nb_0^d)^2} \sum_{\substack{j=1 \\ j \neq i}}^n \sum_{\substack{k=1 \\ k \neq j, i}}^n K_0 \left( \frac{X_k - X_i}{b_0} \right) K_0 \left( \frac{X_k - X_j}{b_0} \right).$$

Moreover, using Lemma 4.1 and some technical details, it can be shown that

$$\sum_{i=1}^n \frac{\mathbb{1}(X_i \in \mathcal{X}_0) g_{in}}{\widehat{g}_{in}} = O_{\mathbb{P}}(1), \quad \sum_{i=1}^n \frac{\mathbb{1}(X_i \in \mathcal{X}_0) \widetilde{g}_{in}}{\widehat{g}_{in}^k} = O_{\mathbb{P}}(1), \quad k = 1, 2.$$

Substituting these orders and (5.5) in (5.4), yields, for  $b_1$  small enough,

$$\text{Var}_n \left[ \sum_{i=1}^n \Sigma_{in} K_1^{(1)} \left( \frac{\varepsilon_i - e}{b_1} \right) \right] = O_{\mathbb{P}} \left( \frac{b_1}{b_0^d} + \frac{b_1^4}{b_0^d} + nb_1^4 \right) = O_{\mathbb{P}} \left( \frac{b_1}{b_0^d} + nb_1^4 \right).$$

Finally, this order, (5.3) and the Markov inequality give

$$\sum_{i=1}^n \Sigma_{in} K_1^{(1)} \left( \frac{\varepsilon_i - e}{b_1} \right) = O_{\mathbb{P}} \left( \frac{b_1}{b_0^d} + nb_1^4 \right)^{1/2}. \square$$



### Proof of Proposition 3.3

Observe that Lemma 4.2 yields that  $\zeta_{in}$  and  $\zeta_{jn}$  are independent given  $X_1, \dots, X_n$  for some  $C > 0$  such that

$\|X_i - X_j\| \geq Cb_0$ . Therefore

$$\text{Var}_n \left( \sum_{i=1}^n \zeta_{in} \right) = \sum_{i=1}^n \text{Var}_n (\zeta_{in}) + \sum_{i=1}^n \sum_{\substack{j=1 \\ j \neq i}}^n \mathbb{1}(\|X_i - X_j\| < Cb_0) \text{Cov}_n (\zeta_{in}, \zeta_{jn}). \quad (5.6)$$

Let  $\mathbb{E}_{in}[\cdot] = \mathbb{E}_n[\cdot | X_1, \dots, X_n, \varepsilon_k, k \neq i]$ . Since  $\hat{m}_{in} - m(X_i)$  depends only on  $(X_1, \dots, X_n, \varepsilon_k, k \neq i)$ ,

$$\sum_{i=1}^n \text{Var}_n (\zeta_{in}) \leq \sum_{i=1}^n \mathbb{E}_n [\zeta_{in}^2] = \sum_{i=1}^n \mathbb{E}_n \left[ \mathbb{1}(X_i \in \mathcal{X}_0) (\hat{m}_{in} - m(X_i))^4 \mathbb{E}_{in} \left[ K_1^{(2)} \left( \frac{\varepsilon_i - e}{b_1} \right)^2 \right] \right],$$

with, using by Lemma 4.4-(4.1),

$$\mathbb{E}_{in} \left[ K_1^{(2)} \left( \frac{\varepsilon_i - e}{b_1} \right)^2 \right] = \int K_1^{(2)} \left( \frac{\epsilon - e}{b_1} \right)^2 f(\epsilon) d\epsilon \leq Cb_1.$$

Therefore Lemma 4.3 implies that

$$\begin{aligned} \sum_{i=1}^n \text{Var}_n (\zeta_{in}) &\leq Cb_1 \sum_{i=1}^n \mathbb{E}_n [\mathbb{1}(X_i \in \mathcal{X}_0) (\hat{m}_{in} - m(X_i))^4] \\ &= O_{\mathbb{P}}(nb_1) \left( b_0^4 + \frac{1}{nb_0^d} \right)^2. \end{aligned}$$

For the sum of the conditional covariances of (5.6), the order is derived from the following equalities:

$$\sum_{i=1}^n \sum_{\substack{j=1 \\ j \neq i}}^n \mathbb{1}(\|X_i - X_j\| < Cb_0) \mathbb{E}_n [\zeta_{in}] \mathbb{E}_n [\zeta_{jn}] = O_{\mathbb{P}}(n^2 b_0^d b_1^6) \left( b_0^4 + \frac{1}{nb_0^d} \right)^2, \quad (5.7)$$

$$\sum_{i=1}^n \sum_{\substack{j=1 \\ j \neq i}}^n \mathbb{1}(\|X_i - X_j\| < Cb_0) \mathbb{E}_n [\zeta_{in} \zeta_{jn}] = O_{\mathbb{P}}(n^2 b_0^d b_1^{7/2}) \left( b_0^4 + \frac{1}{nb_0^d} \right)^2. \quad (5.8)$$

Indeed, since  $b_1$  goes to 0 under (A<sub>9</sub>), the equalities above ensure that

$$\begin{aligned} &\sum_{i=1}^n \sum_{\substack{j=1 \\ j \neq i}}^n \mathbb{1}(\|X_i - X_j\| < Cb_0) \text{Cov}_n (\zeta_{in}, \zeta_{jn}) \\ &= O_{\mathbb{P}} \left[ (n^2 b_0^d b_1^6) \left( b_0^4 + \frac{1}{nb_0^d} \right)^2 + (n^2 b_0^d b_1^{7/2}) \left( b_0^4 + \frac{1}{nb_0^d} \right)^2 \right] \\ &= O_{\mathbb{P}}(n^2 b_0^d b_1^{7/2}) \left( b_0^4 + \frac{1}{nb_0^d} \right)^2. \end{aligned}$$

Combining this with the inequality above and (5.6), and applying the (conditional) Markov inequality, gives the desired result of the lemma. Hence, it remains to prove (5.7) and (5.8). To this end, note that by Lemma 4.4-(4.2) and the Cauchy-Schwartz inequality we have

$$\begin{aligned}
|\mathbb{E}_n [\zeta_{in}]| &= \left| \mathbb{E}_n \left[ \mathbb{1}(X_i \in \mathcal{X}_0) (\widehat{m}_{in} - m(X_i))^2 \mathbb{E}_{in} \left[ K_1^{(2)} \left( \frac{\varepsilon_i - e}{b_1} \right) \right] \right] \right| \\
&= \left| \int K_1^{(2)} \left( \frac{\epsilon - e}{b_1} \right) f(\epsilon) d\epsilon \right| \times \mathbb{E}_n \left[ \mathbb{1}(X_i \in \mathcal{X}_0) (\widehat{m}_{in} - m(X_i))^2 \right] \\
&\leq C b_1^3 \left( \mathbb{E}_n \left[ \mathbb{1}(X_i \in \mathcal{X}_0) (\widehat{m}_{in} - m(X_i))^4 \right] \right)^{1/2},
\end{aligned}$$

uniformly in  $i$ , so that (by Lemma 4.3)

$$\begin{aligned}
\sup_{1 \leq i, j \leq n} |\mathbb{E}_n [\zeta_{in}] \mathbb{E}_n [\zeta_{jn}]| &\leq C b_1^6 \sup_{1 \leq i \leq n} \mathbb{E}_n \left[ \mathbb{1}(X_i \in \mathcal{X}_0) (\widehat{m}_{in} - m(X_i))^4 \right] \\
&= O_{\mathbb{P}}(b_1^6) \left( b_0^4 + \frac{1}{n b_0^d} \right)^2.
\end{aligned}$$

Therefore, since

$$\sum_{i=1}^n \sum_{\substack{j=1 \\ j \neq i}}^n \mathbb{1}(\|X_i - X_j\| < C b_0) = O_{\mathbb{P}}(n^2 b_0^d),$$

this gives

$$\sum_{i=1}^n \sum_{\substack{j=1 \\ j \neq i}}^n \mathbb{1}(\|X_i - X_j\| < C b_0) \mathbb{E}_n [\zeta_{in}] \mathbb{E}_n [\zeta_{jn}] = O_{\mathbb{P}}(n^2 b_0^d b_1^6) \left( b_0^4 + \frac{1}{n b_0^d} \right)^2,$$

which proves (5.7).

For (5.8), let  $\beta_{in}$  and  $\Sigma_{in}$  be as in the statement of Proposition 3.1 and Proposition 3.2 respectively, and define  $Z_{in} = \mathbb{1}(X_i \in \mathcal{X}_0) (\widehat{m}_{in} - m(X_i))^2$ . This gives  $Z_{in} = (\beta_{in} + \Sigma_{in})^2$ , so that, for any  $i \neq j$ ,

$$\mathbb{E}_n [\zeta_{in} \zeta_{jn}] = \mathbb{E}_n \left[ Z_{in} K_1^{(2)} \left( \frac{\varepsilon_j - e}{b_1} \right) \mathbb{E}_{in} \left[ Z_{jn} K_1^{(2)} \left( \frac{\varepsilon_i - e}{b_1} \right) \right] \right], \quad (5.9)$$

where

$$\begin{aligned}
&\mathbb{E}_{in} \left[ Z_{jn} K_1^{(2)} \left( \frac{\varepsilon_i - e}{b_1} \right) \right] \\
&= \beta_{jn}^2 \mathbb{E}_{in} \left[ K_1^{(2)} \left( \frac{\varepsilon_i - e}{b_1} \right) \right] + 2\beta_{jn} \mathbb{E}_{in} \left[ \Sigma_{jn} K_1^{(2)} \left( \frac{\varepsilon_i - e}{b_1} \right) \right] + \mathbb{E}_{in} \left[ \Sigma_{jn}^2 K_1^{(2)} \left( \frac{\varepsilon_i - e}{b_1} \right) \right]. \quad (5.10)
\end{aligned}$$

By Lemma 4.4-(4.2), the first term above gives

$$\left| \beta_{jn}^2 \mathbb{E}_{in} \left[ K_1^{(2)} \left( \frac{\varepsilon_i - e}{b_1} \right) \right] \right| \leq C b_1^3 \beta_{jn}^2. \quad (5.11)$$

Under (A<sub>4</sub>), the second term of (5.10) equals

$$\begin{aligned} & \frac{2\beta_{jn}}{nb_0^d \widehat{g}_{jn}} \sum_{k=1, k \neq j}^n K_0 \left( \frac{X_k - X_j}{b_0} \right) \mathbb{E}_{in} \left[ \varepsilon_k K_1^{(2)} \left( \frac{\varepsilon_i - e}{b_1} \right) \right] \\ &= \frac{2\beta_{jn}}{nb_0^d \widehat{g}_{jn}} K_0 \left( \frac{X_i - X_j}{b_0} \right) \mathbb{E}_{in} \left[ \varepsilon_i K_1^{(2)} \left( \frac{\varepsilon_i - e}{b_1} \right) \right]. \end{aligned}$$

Therefore, since  $K_0$  is bounded under (A<sub>6</sub>), we have (using Lemma 4.4-(4.2))

$$\left| 2\beta_{jn} \mathbb{E}_{in} \left[ \Sigma_{jn} K_1^{(2)} \left( \frac{\varepsilon_i - e}{b_1} \right) \right] \right| \leq \frac{C b_1^3 |\beta_{jn}|}{nb_0^d \widehat{g}_{jn}}. \quad (5.12)$$

For the last term of (5.10), write

$$\begin{aligned} & \mathbb{E}_{in} \left[ \Sigma_{jn}^2 K_1^{(2)} \left( \frac{\varepsilon_i - e}{b_1} \right) \right] \\ &= \frac{\mathbb{1}(X_j \in \mathcal{X}_0)}{(nb_0^d \widehat{g}_{jn})^2} \sum_{\substack{k=1 \\ k \neq j}}^n \sum_{\substack{\ell=1 \\ \ell \neq j}}^n K_0 \left( \frac{X_k - X_j}{b_0} \right) K_0 \left( \frac{X_\ell - X_j}{b_0} \right) \mathbb{E}_{in} \left[ \varepsilon_k \varepsilon_\ell K_1^{(2)} \left( \frac{\varepsilon_i - e}{b_1} \right) \right] \\ &= \frac{\mathbb{1}(X_j \in \mathcal{X}_0)}{(nb_0^d \widehat{g}_{jn})^2} \sum_{k=1, k \neq j}^n K_0^2 \left( \frac{X_k - X_j}{b_0} \right) \mathbb{E}_{in} \left[ \varepsilon_k^2 K_1^{(2)} \left( \frac{\varepsilon_i - e}{b_1} \right) \right]. \end{aligned}$$

Since

$$\begin{aligned} & \left| \mathbb{E}_{in} \left[ \varepsilon_k^2 K_1^{(2)} \left( \frac{\varepsilon_i - e}{b_1} \right) \right] \right| \\ & \leq \max \left\{ \sup_{e' \in \mathbb{R}} \left| \mathbb{E}_{in} \left[ \varepsilon^2 K_1^{(2)} \left( \frac{\varepsilon - e'}{b_1} \right) \right] \right|, \mathbb{E}[\varepsilon^2] \sup_{e' \in \mathbb{R}} \left| \mathbb{E}_{in} \left[ K_1^{(2)} \left( \frac{\varepsilon - e'}{b_1} \right) \right] \right| \right\} \\ & \leq C b_1^3, \end{aligned}$$

uniformly in  $i$ , this gives

$$\left| \mathbb{E}_{in} \left[ \Sigma_{jn}^2 K_1^{(2)} \left( \frac{\varepsilon_i - e}{b_1} \right) \right] \right| \leq \frac{C b_1^3 \mathbb{1}(X_j \in \mathcal{X}_0)}{(nb_0^d \widehat{g}_{jn})^2} \sum_{k=1, k \neq j}^n K_0^2 \left( \frac{X_k - X_j}{b_0} \right).$$

Substituting this, (5.12) and (5.11) in (5.10), yields

$$\left| \mathbb{E}_{in} \left[ Z_{jn} K_1^{(2)} \left( \frac{\varepsilon_i - e}{b_1} \right) \right] \right| \leq C b_1^3 M_n,$$

where

$$M_n = \sup_{1 \leq j \leq n} \left[ \beta_{jn}^2 + \frac{|\beta_{jn}|}{nb_0^d \widehat{g}_{jn}} + \frac{\mathbb{1}(X_j \in \mathcal{X}_0)}{(nb_0^d \widehat{g}_{jn})^2} \sum_{k=1, k \neq j}^n K_0^2 \left( \frac{X_k - X_j}{b_0} \right) \right].$$

Hence from (5.9), the Cauchy-Schwarz inequality and Lemmas 4.3, 4.4 we deduce

$$\begin{aligned} & \sum_{i=1}^n \sum_{\substack{j=1 \\ j \neq i}}^n \mathbb{1}(\|X_i - X_j\| < Cb_0) |\mathbb{E}_n[\zeta_{in} \zeta_{jn}]| \\ & \leq Cb_1^3 M_n \sum_{i=1}^n \sum_{\substack{j=1 \\ j \neq i}}^n \mathbb{1}(\|X_i - X_j\| < Cb_0) \mathbb{E}_n \left| Z_{in} K_1^{(2)} \left( \frac{\varepsilon_j - e}{b_1} \right) \right| \\ & \leq Cb_1^3 M_n \sum_{i=1}^n \sum_{\substack{j=1 \\ j \neq i}}^n \mathbb{1}(\|X_i - X_j\| < Cb_0) \mathbb{E}_n^{1/2} [\mathbb{1}(X_i \in \mathcal{X}_0) (\widehat{m}_{in} - m(X_i))^4] \mathbb{E}_n^{1/2} \left[ K_1^{(2)} \left( \frac{\varepsilon_j - e}{b_1} \right)^2 \right] \\ & = b_1^{7/2} M_n O_{\mathbb{P}} \left( b_0^4 + \frac{1}{nb_0^d} \right) \sum_{i=1}^n \sum_{\substack{j=1 \\ j \neq i}}^n \mathbb{1}(\|X_i - X_j\| \leq Cb_0). \end{aligned}$$

Moreover, using Proposition 3.1 (which gives  $\sup_i |\beta_{in}| = O_{\mathbb{P}}(b_0^2)$ ), Lemma 4.1 and some technical details, it can be shown that

$$M_n = O_{\mathbb{P}} \left( b_0^4 + \frac{b_0^2}{nb_0^d} + \frac{1}{nb_0^d} \right) = O_{\mathbb{P}} \left( b_0^4 + \frac{1}{nb_0^d} \right).$$

Substituting this order in the inequality above, yields (5.8) and finishes the proof of the proposition.  $\square$

### Proof of Proposition 3.4

Observe that by Lemma 4.2, we have

$$\text{Var}_n \left( \sum_{i=1}^n R_{in} \right) = \sum_{i=1}^n \text{Var}_n(R_{in}) + \sum_{i=1}^n \sum_{\substack{j=1 \\ j \neq i}}^n \mathbb{1}(\|X_i - X_j\| < Cb_0) \text{Cov}_n(R_{in}, R_{jn}). \quad (5.13)$$

Let  $\mathbb{E}_{in}[\cdot] = \mathbb{E}_n[\cdot | X_1, \dots, X_n, \varepsilon_k, k \neq i]$ , and write

$$\sum_{i=1}^n \text{Var}_n(R_{in}) \leq \sum_{i=1}^n \mathbb{E}_n[R_{in}^2] = \sum_{i=1}^n \mathbb{E}_n \left[ \mathbb{1}(X_i \in \mathcal{X}_0) (\widehat{m}_{in} - m(X_i))^6 \mathbb{E}_{in}[I_{in}^2] \right],$$

with, using (A<sub>4</sub>), the Cauchy-Schwarz inequality and Lemma 4.4-(4.3),

$$\begin{aligned}
\mathbb{E}_{in} [I_{in}^2] &= \mathbb{E}_{in} \left[ \left\{ \int_0^1 (1-t)^2 K_1^{(3)} \left( \frac{\varepsilon_i - t(\widehat{m}_{in} - m(X_i)) - e}{b_1} \right) dt \right\}^2 \right] \\
&\leq \mathbb{E}_{in} \left[ \int_0^1 (1-t)^4 K_1^{(3)} \left( \frac{\varepsilon_i - t(\widehat{m}_{in} - m(X_i)) - e}{b_1} \right)^2 dt \right] \\
&= \int_0^1 (1-t)^4 \left[ \int K_1^{(3)} \left( \frac{\varepsilon - t(\widehat{m}_{in} - m(X_i)) - e}{b_1} \right)^2 f(\varepsilon) d\varepsilon \right] dt \leq Cb_1.
\end{aligned}$$

Therefore Lemma 4.3 implies that

$$\begin{aligned}
\sum_{i=1}^n \text{Var}_n(R_{in}) &\leq Cnb_1 \sup_{1 \leq i \leq n} \mathbb{E}_n [\mathbb{1}(X_i \in \mathcal{X}_0) (\widehat{m}_{in} - m(X_i))^6] \\
&= O_{\mathbb{P}}(nb_1) \left( b_0^4 + \frac{1}{nb_0^d} \right)^3.
\end{aligned} \tag{5.14}$$

For the second term of (5.13), write

$$\begin{aligned}
|\text{Cov}_n(R_{in}, R_{jn})| &\leq (\text{Var}_n(R_{in}) \text{Var}_n(R_{jn}))^{1/2} \\
&\leq Cb_1 \sup_{1 \leq i \leq n} \mathbb{E}_n [\mathbb{1}(X_i \in \mathcal{X}_0) (\widehat{m}_{in} - m(X_i))^6] = O_{\mathbb{P}}(b_1) \left( b_0^4 + \frac{1}{nb_0^d} \right)^3,
\end{aligned}$$

uniformly in  $i$  and  $j$ . Hence

$$\begin{aligned}
&\sum_{i=1}^n \sum_{\substack{j=1 \\ j \neq i}}^n (\|X_i - X_j\| \leq Cb_0) |\text{Cov}_n(R_{in}, R_{jn})| \\
&= O_{\mathbb{P}}(b_1) \left( b_0^4 + \frac{1}{nb_0^d} \right)^3 \sum_{i=1}^n \sum_{\substack{j=1 \\ j \neq i}}^n (\|X_i - X_j\| \leq Cb_0) \\
&= O_{\mathbb{P}}(b_1) \left( b_0^4 + \frac{1}{nb_0^d} \right)^3 (n^2 b_0^d).
\end{aligned}$$

Finally, this order, (5.14) and (5.13) give, since  $nb_0^d$  diverges under (A<sub>8</sub>),

$$\text{Var} \left( \sum_{i=1}^n R_{in} \right) = O_{\mathbb{P}}(nb_1 + n^2 b_0^d b_1) \left( b_0^4 + \frac{1}{nb_0^d} \right)^3 = O_{\mathbb{P}}(n^2 b_0^d b_1) \left( b_0^4 + \frac{1}{nb_0^d} \right)^3. \square$$

## Appendix: Proof of the intermediate results

### Proof of Lemma 4.1

First note that by  $(A_7)$ , we have  $\int z K_0(z) dz = 0$  and  $\int K_0(z) dz = 1$ . Therefore  $(A_1)$ ,  $(A_2)$  and the second-order Taylor expansion, yield, for  $b_0$  small enough and any  $x$  in  $\mathcal{X}_0$ ,

$$\begin{aligned} |\bar{g}_n(x) - g(x)| &= \left| \frac{1}{b_0^d} \int K_0\left(\frac{z-x}{b_0}\right) g(z) dz - g(x) \right| = \left| \int K_0(z) [g(x + b_0 z) - g(x)] dz \right| \\ &= \left| \int K_0(z) \left[ b_0 g^{(1)}(x) z + \frac{b_0^2}{2} z g^{(2)}(x + \theta b_0 z) z^\top \right] dz \right|, \quad \theta = \theta(x, b_0 z) \in [0, 1] \\ &= \frac{b_0^2}{2} \left| \int z g^{(2)}(x + \theta b_0 z) z^\top K_0(z) dz \right| \leq C b_0^2, \end{aligned}$$

so that

$$\sup_{x \in \mathcal{X}_0} |\bar{g}_n(x) - g(x)| = O(b_0^2).$$

This gives the first result of the lemma. To prove the second and third results of the lemma, note that it is sufficient to show that

$$\sup_{x \in \mathcal{X}_0} |\hat{g}_n(x) - \bar{g}_n(x)| = O_{\mathbb{P}} \left( \frac{\ln n}{n b_0^d} \right)^{1/2},$$

since  $\bar{g}_n(x)$  is asymptotically bounded away from 0 over  $\mathcal{X}_0$  and that  $|\bar{g}_n(x) - g(x)| = O(b_0^2)$  uniformly for  $x$  in  $\mathcal{X}_0$ . This follows from Theorem 1 in Einmahl and Mason (2005).  $\square$

### Proof of Lemma 4.2

Since  $K_0(\cdot)$  has a compact support under  $(A_6)$ , there is a  $C > 0$  such that  $\|X_i - X_j\| \geq C b_0$  implies that for any integer number  $k$  of  $[1, n]$ ,  $K_0((X_k - X_i)/b_0) = 0$  if  $K_0((X_j - X_k)/b_0) \neq 0$ . Let  $D_j \subset [1, n]$  be such that an integer number  $k$  of  $[1, n]$  is in  $D_j$  if and only if  $K_0((X_j - X_k)/b_0) \neq 0$ . Abbreviate  $\mathbb{P}(\cdot | X_1, \dots, X_n)$  into  $\mathbb{P}_n$  and assume that  $\|X_i - X_j\| \geq C b_0$  so that  $D_i$  and  $D_j$  have an empty intersection. Note also that taking  $C$  large enough ensures that  $i$  is not in  $D_j$  and  $j$  is not in  $D_i$ . It then follows, under  $(A_4)$  and since  $D_i$  and

$D_j$  only depend upon  $X_1, \dots, X_n$ ,

$$\begin{aligned}
& \mathbb{P}_n \left( (\widehat{m}_{in} - m(X_i), \varepsilon_i) \in A \text{ and } (\widehat{m}_{jn} - m(X_j), \varepsilon_j) \in B \right) \\
&= \mathbb{P}_n \left( \left( \frac{\sum_{k \in D_i \setminus \{i\}} (m(X_k) - m(X_i) + \varepsilon_k) K_0((X_k - X_i)/b_0)}{\sum_{k \in D_i \setminus \{i\}} K_0((X_k - X_i)/b_0)}, \varepsilon_i \right) \in A \right. \\
&\quad \left. \text{and } \left( \frac{\sum_{\ell \in D_j \setminus \{j\}} (m(X_\ell) - m(X_j) + \varepsilon_\ell) K_0((X_\ell - X_j)/b_0)}{\sum_{\ell \in D_j \setminus \{j\}} K_0((X_\ell - X_j)/b_0)}, \varepsilon_j \right) \in B \right) \\
&= \mathbb{P}_n \left( \left( \frac{\sum_{k \in D_i \setminus \{i\}} (m(X_k) - m(X_i) + \varepsilon_k) K_0((X_k - X_i)/b_0)}{\sum_{k \in D_i \setminus \{i\}} K_0((X_k - X_i)/b_0)}, \varepsilon_i \right) \in A \right) \\
&\quad \times \mathbb{P}_n \left( \left( \frac{\sum_{\ell \in D_j \setminus \{j\}} (m(X_\ell) - m(X_j) + \varepsilon_\ell) K_0((X_\ell - X_j)/b_0)}{\sum_{\ell \in D_j \setminus \{j\}} K_0((X_\ell - X_j)/b_0)}, \varepsilon_j \right) \in B \right) \\
&= \mathbb{P}_n((\widehat{m}_{in} - m(X_i), \varepsilon_i) \in A) \times \mathbb{P}_n((\widehat{m}_{jn} - m(X_j), \varepsilon_j) \in B).
\end{aligned}$$

This gives the result of Lemma 4.2, since both  $(\widehat{m}_{in} - m(X_i), \varepsilon_i)$  and  $(\widehat{m}_{jn} - m(X_j), \varepsilon_j)$  are independent given  $X_1, \dots, X_n$ .  $\square$

### Proof of Lemma 4.3

Let  $\beta_{in}$  as in the statement of Proposition 3.1 and set

$$g_{in} = \frac{1}{nb_0^d} \sum_{j=1, j \neq i}^n K_0^4 \left( \frac{X_j - X_i}{b_0} \right), \quad \widetilde{g}_{in} = \frac{1}{nb_0^d} \sum_{j=1, j \neq i}^n K_0^2 \left( \frac{X_j - X_i}{b_0} \right).$$

The proof of the lemma is based on the following bound:

$$\mathbb{E}_n \left[ \mathbf{1}(X_i \in \mathcal{X}_0) (\widehat{m}_{in} - m(X_i))^k \right] \leq C \left[ \beta_{in}^k + \frac{\mathbf{1}(X_i \in \mathcal{X}_0) \widetilde{g}_{in}^{k/2}}{(nb_0^d)^{(k/2)} \widehat{g}_{in}^k} \right], \quad k \in \{4, 6\}. \quad (1)$$

Indeed, taking successively  $k = 4$  and  $k = 6$  in (1), we have, by (5.1) and Lemma 4.1

$$\begin{aligned}
\sup_{1 \leq i \leq n} \mathbb{E}_n \left[ \mathbf{1}(X_i \in \mathcal{X}_0) (\widehat{m}_{in} - m(X_i))^4 \right] &= O_{\mathbb{P}} \left( b_0^8 + \frac{1}{(nb_0^d)^2} \right) = O_{\mathbb{P}} \left( b_0^4 + \frac{1}{nb_0^d} \right)^2, \\
\sup_{1 \leq i \leq n} \mathbb{E}_n \left[ \mathbf{1}(X_i \in \mathcal{X}_0) (\widehat{m}_{in} - m(X_i))^6 \right] &= O_{\mathbb{P}} \left( b_0^{12} + \frac{1}{(nb_0^d)^3} \right) = O_{\mathbb{P}} \left( b_0^4 + \frac{1}{nb_0^d} \right)^3,
\end{aligned}$$

which gives the desired results. Hence it remains to prove (1). To this end, let  $\Sigma_{in}$  be as in the statement of Proposition 3.2 and observe that  $\mathbf{1}(X_i \in \mathcal{X}_0) (\widehat{m}_{in} - m(X_i)) = \beta_{in} + \Sigma_{in}$ . Since  $\beta_{in}$  depends only on

$(X_1, \dots, X_n)$ , this gives, for any  $k \in \{4, 6\}$ , we have

$$\mathbb{E}_n \left[ \mathbb{1}(X_i \in \mathcal{X}_0) (\widehat{m}_{in} - m(X_i))^k \right] \leq C \beta_{in}^k + C \mathbb{E}_n [\Sigma_{in}^k]. \quad (2)$$

The order of the second term of (2) is computed by applying Theorem 2 in Whittle (1960) or the Marcinkiewicz-Zygmund inequality (see e.g Chow and Teicher, 2003, p. 386). These inequalities show that for linear form  $L = \sum_{j=1}^n a_j \zeta_j$  with independent mean-zero random variables  $\zeta_1, \dots, \zeta_n$ , it holds that, for any  $k \geq 1$ ,

$$\mathbb{E} |L^k| \leq C(k) \left[ \sum_{j=1}^n a_j^2 \mathbb{E}^{2/k} |\zeta_j^k| \right]^{k/2},$$

where  $C(k)$  is a positive real depending only on  $k$ . Now, observe that for any integer  $i \in [1, n]$ ,

$$\Sigma_{in} = \sum_{j=1, j \neq i}^n \sigma_{jin}, \quad \sigma_{jin} = \frac{\mathbb{1}(X_i \in \mathcal{X}_0)}{nb_0^d \widehat{g}_{in}} \varepsilon_j K_0 \left( \frac{X_j - X_i}{b_0} \right).$$

Since under  $(A_4)$ , the  $\sigma_{jin}$ 's, ( $j = 1, \dots, n$ ), are centered independent variables given  $X_1, \dots, X_n$ , this yields, for any  $k \in \{4, 6\}$ ,

$$\mathbb{E}_n [\Sigma_{in}^k] \leq C \mathbb{E} [\varepsilon^k] \left[ \frac{\mathbb{1}(X_i \in \mathcal{X}_0)}{(nb_0^d)^2 \widehat{g}_{in}^2} \sum_{j=1}^n K_0^2 \left( \frac{X_j - X_i}{b_0} \right) \right]^{k/2} \leq \frac{C \mathbb{1}(X_i \in \mathcal{X}_0) \widehat{g}_{in}^{k/2}}{(nb_0^d)^{(k/2)} \widehat{g}_{in}^k}.$$

Hence, this bound and (2) imply that

$$\mathbb{E}_n \left[ \mathbb{1}(X_i \in \mathcal{X}_0) (\widehat{m}_{in} - m(X_i))^k \right] \leq C \left[ \beta_{in}^k + \frac{\mathbb{1}(X_i \in \mathcal{X}_0) \widehat{g}_{in}^{k/2}}{(nb_0^d)^{(k/2)} \widehat{g}_{in}^k} \right],$$

which proves (1), and then completes the proof of the lemma.  $\square$

## Proof of Lemma 4.4

Set  $h_p(e) = e^p f(e)$ ,  $p \in [0, 2]$ . For the first inequality of (4.1), note that under  $(A_5)$  and  $(A_7)$ , the change of variable  $\epsilon = e + b_1 v$  give, for any  $\ell \in \{1, 2, 3\}$ ,

$$\begin{aligned} \left| \int K_1^{(\ell)} \left( \frac{\epsilon - e}{b_1} \right)^2 \epsilon^p f(\epsilon) d\epsilon \right| &= \left| b_1 \int K_1^{(\ell)}(v)^2 h_p(e + b_1 v) dv \right| \\ &\leq b_1 \sup_{e \in \mathbb{R}} |h_p(e)| \int K_1^{(\ell)}(v)^2 dv \\ &\leq C b_1, \end{aligned} \quad (3)$$



which yields the first inequality of (4.1). For the second inequality of (4.1), observe that under (A<sub>7</sub>) we have  $\int K_1^{(\ell)}(v)dv = 0$ . Therefore, since  $h_p(\cdot)$  has bounded second order derivatives under (A<sub>5</sub>), the Taylor inequality gives

$$\begin{aligned} \left| \int K_1^{(\ell)} \left( \frac{\epsilon - e}{b_1} \right) \epsilon^p f(\epsilon) d\epsilon \right| &= b_1 \left| \int K_1^{(\ell)}(v) [h_p(e + b_1 v) - h_p(e)] dv \right| \\ &\leq b_1^2 \sup_{e \in \mathbb{R}} |h_p^{(1)}(e)| \int |v K_1^{(\ell)}(v)| dv \leq C b_1^2. \end{aligned}$$

which completes the proof of (4.1). The first inequalities of (4.2) and (4.3) are given by (3). The second inequalities of (4.2) and (4.3) are proved simultaneously. For this, note that for any integer  $\ell \in \{2, 3\}$ ,

$$\int K_1^{(\ell)} \left( \frac{\epsilon - e}{b_1} \right) h_p(\epsilon) d\epsilon = b_1 \int K_1^{(\ell)}(v) h_p(e + b_1 v) dv.$$

By (A<sub>7</sub>),  $K_1$  has a compact support and satisfies  $\int K_1^{(\ell)}(v)dv = 0$  and  $\int v K_1^{(\ell)}(v)dv = 0$ . Hence the second order Taylor expansion applied to  $h_p(\cdot)$  gives, for some  $\theta = \theta(e, b_1, v) \in [0, 1]$ ,

$$\begin{aligned} \left| \int K_1^{(\ell)} \left( \frac{\epsilon - e}{b_1} \right) h_p(\epsilon) d\epsilon \right| &= \left| b_1 \int K_1^{(\ell)}(v) [h_p(e + b_1 v) - h_p(e)] dv \right| \\ &= \left| b_1 \int K_1^{(\ell)}(v) \left[ b_1 v h_p^{(1)}(e) + \frac{b_1^2 v^2}{2} h_p^{(2)}(e + \theta b_1 v) \right] dv \right| \\ &= \left| \frac{b_1^3}{2} \int v^2 K_1^{(\ell)}(v) h_p^{(2)}(e + \theta b_1 v) dv \right| \\ &\leq \frac{b_1^3}{2} \sup_{e \in \mathbb{R}} |h_p^{(2)}(e)| \int |v^2 K_1^{(\ell)}(v)| dv \leq C b_1^3. \square \end{aligned}$$

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